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ON A PRODUCT FORMULA FOR UNITARY GROUPS

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Abstract

For any nonnegative self-adjoint operators A and B in a separable Hilbert space, the Trotter-type formula $[(e^{i2tA/n} + e^{i2tB/n})/2]^n$ is shown to converge strongly in the norm closure of $\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})$ for some subsequence and for almost every $t \in \mathbb{R}$. This result extends to the degenerate case, and to Kato-functions following the method of T. Kato (see 'Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroup', *Topics in functional analysis* (ed. M. Kac, Academic Press, New York, 1978) 185–195). Moreover, the restrictions on the convergence can be removed by considering functions other than the exponential.

In a famous paper [**7**], T. Kato proved that for any nonnegative self-adjoint operators A and B in a Hilbert space $\mathcal{H}$, the Trotter product formula $(e^{-tA/n}e^{-tB/n})^n$ converges strongly to the (degenerate) semigroup generated by the form-sum $A\dot{+}B$ for any t with $\mathrm{Re}\, t > 0$. For real $t > 0$, he also enlarged the result to a class of so-called *Kato-functions* (see (1.2) below), and to degenerate semigroups. However, the convergence on the boundary $i\mathbb{R}$ remains an unclear problem in this generality [**2**, **4**]. For functions f such that $\mathrm{Im}\, f \leqslant 0$ (for example, $f(s) = (1 + is)^{-1}$), Lapidus found such an extension [**8**], but it does not apply to unitary groups: that is, the imaginary exponential function $f(s) = e^{is}$.

1. *Statement of the result*

Since this note is closely related to Kato's paper [**7**], it is convenient to use similar notation. Let A and B denote nonnegative self-adjoint operators defined in closed subspaces M_A and M_B of a separable Hilbert space $\mathcal{H}$, and let P_A and P_B denote the orthogonal projections on M_A and M_B. Let $\mathcal{D}' = \mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})$, let $\mathcal{H}'$ be the closure of $\mathcal{D}'$, and let P' be the orthogonal projection on $\mathcal{H}'$. Note that $\mathcal{D}'$ is not necessarily dense; in fact, it may reduce to $\{0\}$. The form-sum $C = A\dot{+}B$ is defined as the self-adjoint operator in $\mathcal{H}'$ associated with the nonnegative, closed quadratic form $u \mapsto \|A^{1/2}u\|^2 + \|B^{1/2}u\|^2$, $u \in \mathcal{D}'$. We consider Trotter-type product formulae $F(t/n)^n$ based on the arithmetic mean

$$F(t) = \frac{f(2tA)P_A + g(2tB)P_B}{2}. \tag{1.1}$$

The functions f and g are assumed, first, to satisfy Kato's conditions [**7**]: they are Borel measurable on $[0, \infty)$ with

$$f(0) = 1, \qquad f'(+0) = -1, \qquad 0 \leqslant f(t) \leqslant 1,\ t > 0 \tag{1.2}$$

(and the same for g). Moreover, they admit bounded holomorphic extensions to $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ with $|f(z)| \leqslant 1$, $|g(z)| \leqslant 1$. For simplicity, we also assume

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that f and g have continuous extensions to the imaginary axis. Examples of such functions are: $z \mapsto e^{-z}$ or $z \mapsto (1 + z/k)^{-k}$ for any $k > 0$.

By the functional calculus for normal operators, $F(z)$ is well defined for $\operatorname{Re} z \geqslant 0$, and is bounded in operator-norm by 1. Moreover, $z \mapsto F(z)$ is a holomorphic operator-valued function in $\mathbb{C}_+$, which follows from the same property for $z \mapsto f(zA)$ and $g(zB)$, respectively. Let E^A be the spectral measure associated to the self-adjoint operator A; then

$$f(zA) = \int_0^\infty f(z\lambda) dE_\lambda^A.$$

By observing that

$$|f'(z)| \leqslant (\operatorname{Re} z)^{-1}, \qquad z \in \mathbb{C}_+,$$

we find that the strong complex derivative

$$s - \lim_{h \to 0} \frac{f((z+h)A) - f(zA)}{h} = \int_0^\infty \lambda f'(z\lambda) dE_\lambda^A \tag{1.3}$$

exists as a bounded operator for any $z \in \mathbb{C}_+$, which implies the holomorphy.

THEOREM 1.1. *Let $\mathcal{H}$ be a separable Hilbert space. Let A, M_A, P_A, B, M_B, P_B, C, $\mathcal{H}'$, P', f, g, and F be as defined above. For any $u \in \mathcal{H}$, one has*

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} \phi(t) F(it/n)^n u \, dt = \int_{-\infty}^{+\infty} \phi(t) e^{-itC} P' u \, dt, \qquad \phi \in L^1(\mathbb{R}). \tag{1.4}$$

Moreover, there exist a set $L \subset \mathbb{R}$ with zero Lebesgue measure and an increasing function $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$, such that:

$$\forall u \in \mathcal{H}', \qquad F(it/\varphi(n))^{\varphi(n)} u \longrightarrow e^{-itC} P' u, \qquad t \in \mathbb{R} \setminus L. \tag{1.5}$$

One sees that the strong convergence, valid in the open right half-plane, cannot extend exactly to the boundary $i\mathbb{R}$, as has already been noted in [**4**, **9**] with counter-examples: the strong convergence on the boundary is restricted to the subspace $\mathcal{H}'$. However, the weaker convergence (1.4) has already been observed [**2**, **5**].

2. *Proof*

Let us consider, for $\operatorname{Re} t \geqslant 0$ and $\tau > 0$,

$$S_{t,\tau} = \tau^{-1}(I - F(t\tau)), \tag{2.1}$$

which is a holomorphic operator-valued function of $t \in \mathbb{C}_+$. The main step of the proof is to show that the strong convergence

$$s - \lim_{\tau \to 0} (I + S_{t,\tau})^{-1} = (I + tC)^{-1} P' \tag{2.2}$$

holds for $t \in \mathbb{C}_+$, and remains true for almost all $t \in i\mathbb{R}$ and on some subsequence. This will give the desired result (1.5) for $u \in \mathcal{H}'$, by Chernoff's theorem (see below). The convergence on the boundary is obtained by a useful result of Feldman [**5**, Theorem 5.1], which we state here in a slightly more general form. (Here, $L^1(\mathbb{R}, \mathcal{H})$ denotes the Banach space of Bochner integrable $\mathcal{H}$-valued functions on $\mathbb{R}$; see [**1**, Section 1.1].)

LEMMA 2.1. *Let $\mathcal{H}$ be a separable Hilbert space, and let $\{\Psi_\tau : 0 < \tau < 1\}$ be a uniformly bounded family of bounded holomorphic $\mathcal{H}$-valued functions defined in $\mathbb{C}_+$. Suppose that*

$$\Psi_\tau(z) \stackrel{\tau\to 0}{\longrightarrow} \Psi(z), \qquad \text{for each } z \in \mathbb{C}_+.$$

Then $\Psi_\tau(i\cdot) \stackrel{\tau\to 0}{\longrightarrow} \Psi(i\cdot)$ in the $\sigma(L^\infty, L^1)$ topology on $L^\infty(\mathbb{R},\mathcal{H})$; that is, for each $v \in L^1(\mathbb{R},\mathcal{H})$,

$$\int_{\mathbb{R}} (v(t), \Psi_\tau(it))dt \stackrel{\tau\to 0}{\longrightarrow} \int_{\mathbb{R}} (v(t), \Psi(it))dt. \tag{2.3}$$

Proof. Since $\mathcal{H}$ is separable, the bounded holomorphic functions Ψ_τ have boundary values for almost every $is \in i\mathbb{R}$, and one has the Poisson integral representation for any $t > 0$ and $s \in \mathbb{R}$ (see [**6**, Sections 6.4 and 6.5]):

$$\Psi_\tau(t + is) = \int_{-\infty}^{+\infty} \frac{t/\pi}{t^2 + (s - s')^2} \Psi_\tau(is')ds' = P_t * \Psi_\tau(i\cdot).$$

The kernel P_t is in fact an approximate identity: $P_t * \phi \stackrel{L^1}{\longrightarrow} \phi$ as $t \to 0$; see [**1**, Lemma 1.3.3] for the vector-valued case. Using the identity

$$\int_{\mathbb{R}} (v(s), [P_t * h](s))ds = \int_{\mathbb{R}} ([P_t * v](s), h(s))ds,$$

we obtain the following estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}} (v(s), [\Psi_\tau(is) - \Psi(is)])ds\right| \leqslant &\int_{\mathbb{R}} \|v(s)\| \|\Psi_\tau(t + is) - \Psi(t + is)\| ds \\ &+ \int_{\mathbb{R}} \|v(s) - (P_t * v)(s)\| \|\Psi_\tau(is) - \Psi(is)\| ds.\end{aligned} \tag{2.4}$$

The second term on the right-hand side of (2.4) (that is, the last line) can be made arbitrarily small by choosing t sufficiently small. Also, for any t, the first integral in the right-hand side of (2.4) tends to 0 as $\tau \to 0$, by Lebesgue's theorem. ☐

REMARK 2.2. In fact, a similar argument leads to the stronger statement:

$$s - \lim_{\tau\to 0} \int_{\mathbb{R}} \phi(t)\Psi_\tau(it)dt = \int_{\mathbb{R}} \phi(t)\Psi(it)dt$$

for each (numerical) $\phi \in L^1(\mathbb{R},\mathbb{C})$.

It is convenient to introduce the following bounded operators, for $\mathrm{Re}\, t \geqslant 0$ and $\tau > 0$:

$$A_{t,\tau} = \tau^{-1}[I - f(t\tau A)P_A], \qquad \text{and} \qquad B_{t,\tau} = \tau^{-1}[I - g(t\tau B)P_B]. \tag{2.5}$$

Since $|f|$ and $|g|$ are bounded by 1, one has $\|f(t\tau A)\|$ and $\|g(t\tau B)\|$ equal at most to 1, which implies that $A_{t,\tau}$ and $B_{t,\tau}$ are accretive (that is, their real parts are non-negative).

LEMMA 2.3. *For any $t \in \mathbb{C}_+$, we have $s - \lim_{\tau\to 0}(I + S_{t,\tau})^{-1} = (I + tC)^{-1}P'$. Moreover, for any $v \in L^1(\mathbb{R},\mathcal{H})$, $u \in \mathcal{H}$ and $t \in \mathbb{R}$, one has*

$$\lim_{\tau\to 0} \int_{\mathbb{R}} (v(t), (I + S_{it,\tau})^{-1}u)dt = \int_{\mathbb{R}} (v(t), (I + itC)^{-1}P'u)dt. \tag{2.6}$$

Proof. Since $S_{t,\tau} = A_{t,2\tau} + B_{t,2\tau}$, the strong convergence of $(I+S_{t,\tau})^{-1}$ for $t > 0$ follows from [**7**, Lemmas 2.2 and 2.3]. Then it extends to the open right half-plane by the theorem of Vitali: for any $\tau > 0$, $(I + S_{t,\tau})^{-1}$ is a holomorphic function of t, and is bounded by 1.

The convergence on the boundary (2.6) follows from Lemma 2.1. $\square$

For any fixed $u \in \mathcal{H}$ and $t \in \mathbb{R}$, we set $w_{t,\tau} = (I + S_{it,\tau})^{-1}u$, $\tau > 0$. Then one finds that

$$(u, w_{t,\tau}) = \|w_{t,\tau}\|^2 + (A_{it,2\tau}w_{t,\tau}, w_{t,\tau}) + (B_{it,2\tau}w_{t,\tau}, w_{t,\tau}) \tag{2.7}$$

with

$$\begin{aligned}\operatorname{Re}(A_{it,2\tau}w_{t,\tau}, w_{t,\tau}) &= \|(\operatorname{Re} A_{it,2\tau})^{1/2}w_{t,\tau}\|^2 \geqslant 0;\\ \operatorname{Re}(B_{it,2\tau}w_{t,\tau}, w_{t,\tau}) &= \|(\operatorname{Re} B_{it,2\tau})^{1/2}w_{t,\tau}\|^2 \geqslant 0.\end{aligned}$$

Therefore

$$\|w_{t,\tau}\|^2 \leqslant \operatorname{Re}(u, w_{t,\tau}) \leqslant |(u, w_{t,\tau})| \leqslant \|u\|\|w_{t,\tau}\|,$$

and thus $\|w_{t,\tau}\| \leqslant \|u\|$, $\tau > 0$.

Lemma 2.4. *Let α_n be any sequence of positive numbers with limit zero. There exists a set $L \subset \mathbb{R}$ of zero Lebesgue measure, and a subsequence τ_n of α_n, such that for each $t \in \mathbb{R} \setminus L$, $s - \lim_{n\to\infty}(I + S_{it,\tau_n})^{-1} = (I + itC)^{-1}P'$.*

Proof. It follows from Lemma 2.3 that:

$$\int_{\mathbb{R}} (1 + t^2)^{-1}(u, w_{t,\tau})dt \to \int_{\mathbb{R}} (1 + t^2)^{-1}(u, w_t)dt \qquad \text{with } w_t = (I + itC)^{-1}P'u.$$

Thus the same is true for the real part, and we have, by (2.7),

$$\operatorname{Re}(u, w_{t,\tau}) = \|w_{t,\tau}\|^2 + \|(\operatorname{Re} A_{it,2\tau})^{1/2}w_{t,\tau}\|^2 + \|(\operatorname{Re} B_{it,2\tau})^{1/2}w_{t,\tau}\|^2. \tag{2.8}$$

We observe that $\operatorname{Re}(u, w_t) = \operatorname{Re}((I + itC)w_t, w_t) = \|w_t\|^2$, and that

$$\int_{\mathbb{R}} \operatorname{Re}(w_{t,\tau}, w_t)\frac{dt}{1 + t^2} \xrightarrow{\tau\to 0} \int_{\mathbb{R}} \|w_t\|^2\frac{dt}{1 + t^2}.$$

Then one finds that

$$\int_{\mathbb{R}} \big(\|w_{t,\tau} - w_t\|^2 + \|(\operatorname{Re} A_{it,2\tau})^{1/2}w_{t,\tau}\|^2 + \|(\operatorname{Re} B_{it,2\tau})^{1/2}w_{t,\tau}\|^2\big)\frac{dt}{1 + t^2} \xrightarrow{\tau\to 0} 0;$$

in particular,

$$\int_{\mathbb{R}} \|w_{t,\tau} - w_t\|^2(1 + t^2)^{-1}dt \xrightarrow{\tau\to 0} 0.$$

This means that the functions $t \mapsto \|w_{t,\tau} - w_t\|$ converge to 0 in $L^2(\mathbb{R}, \mu)$ as $\tau \to 0$, with the finite measure $d\mu = (1 + t^2)^{-1}dt$. Let $(e_m)_{m\in\mathbb{N}}$ be a basis of the separable Hilbert space $\mathcal{H}$. For $u = e_1$, the above L^2-convergence implies that there exist $L_1 \subset \mathbb{R}$ with $\mu(L_1) = 0$, and some increasing function $\varphi_1 : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$(I + S_{it,\alpha_{\varphi_1(n)}})^{-1}e_1 \to (I + itC)^{-1}P'e_1, \qquad \text{as } n \to \infty,$$

for any $t \in \mathbb{R} \setminus L_1$. Then, for $u = e_2$, there exist $L_2 \subset \mathbb{R}$ with $\mu(L_2) = 0$, and an increasing function $\varphi_2 : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$(I + S_{it,\alpha_{\varphi_1\circ\varphi_2(n)}})^{-1}e_2 \to (I + itC)^{-1}P'e_2, \qquad \text{as } n \to \infty,$$

for any $t \in \mathbb{R} \setminus L_2$, and so on for each $m \in \mathbb{N}$.

Finally, by the diagonal procedure, we consider the sequence $\tau_n = \alpha_{\varphi_1\circ\ldots\circ\varphi_n(n)}$, and we find that convergence holds for each vector e_m of the basis, and for each $t \in \mathbb{R} \setminus L$, where $L = \bigcup_{m\in\mathbb{N}} L_m$. We have $\mu(L) = 0$, and thus L also has zero Lebesgue measure. Since the operators $(I + S_{it,\tau})^{-1}$ are uniformly bounded, this implies the strong convergence for any vector $u \in \mathcal{H}$. □

Proof of Theorem 1.1. We consider

$$Z_{t,n} = (n/t)[F(it/n) - I] = -t^{-1} S_{it,1/n} \qquad \text{and} \qquad \alpha_n = 1/n.$$

Let L be as in Lemma 2.4, and let $t \in \mathbb{R} \setminus L$, $t \neq 0$. By [**3**, Theorem 3.17] and Lemma 2.4, one obtains, for some increasing function $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$:

$$\lim_{n\to\infty} e^{sZ_{t,\varphi(n)}} u = e^{-isC} P' u, \qquad u \in \mathcal{H}',\ s \in \mathbb{R}. \tag{2.9}$$

By Chernoff's lemma [**3**, Lemmas 3.27 and 3.29], one has

$$\lim_{n\to\infty} \|F(it/\varphi(n))^{\varphi(n)} u - e^{\varphi(n)(F(it/\varphi(n))-I)} u\| = 0, \qquad u \in \mathcal{H}'. \tag{2.10}$$

Thus we obtain the convergence (1.5) in $\mathcal{H}'$.

The weak convergence (1.4) follows by using Kato's result for $\operatorname{Re} t > 0$, together with Remark 2.2. □

Corollary 2.5. *Suppose that the Kato functions f and g are holomorphic and bounded in some half-plane $\Pi_\theta = \{z \in \mathbb{C} : \operatorname{Re}(e^{-i\theta} z) > 0\}$ with $0 < \theta < \pi/2$, and that there exists $\theta' \in (-\pi/2, \pi/2)$, such that $1 - f(\Pi_\theta) \subseteq \Pi_{\theta'}$ and $1 - g(\Pi_\theta) \subseteq \Pi_{\theta'}$. Then one has*

$$\forall u \in \mathcal{H}', \qquad \lim_{n\to\infty} F(it/n)^n u = e^{-itC} P' u, \qquad t > 0.$$

Proof. The holomorphy of, respectively, $z \mapsto f(zA)$ and $z \mapsto g(zB)$ follows from the spectral representation (1.3) and the estimate that $|f'(z)| \leqslant M|\operatorname{Re}(e^{-i\theta} z)|^{-1}$, $z \in \Pi_\theta$, for some $M > 0$. The condition on the ranges of f and g implies that the spectra of the normal operators $A_{z,\tau}$ and $B_{z,\tau}$ lie in the half-plane $\Pi_{\theta'}$ for each $z \in \Pi_\theta$ and $\tau > 0$. It follows that the numerical range of $S_{z,\tau} = A_{z,2\tau} + B_{z,2\tau}$ lies in the same half-plane $\Pi_{\theta'}$, and that $\|(I + S_{z,\tau})^{-1}\|$ is uniformly bounded for $z \in \Pi_\theta$ and $\tau > 0$. Then the convergence (2.2) for any $z \in \Pi_\theta$ follows from [**7**] by the theorem of Vitali. The end of the proof is similar to that of Theorem 1.1. □

Remark 2.6. The above corollary applies to the function $z \mapsto (1+z/k)^{-k}$, but we do not know whether Theorem 1.1 can be improved for the exponential function. The subsequence appearing in Theorem 1.1 makes the result somewhat unsatisfactory. In fact, this restriction is not necessary if we assume that the functions $t \mapsto (I + S_{it,\tau})^{-1}$ are strongly equicontinuous with respect to $\tau > 0$, at some point $t_0 \neq 0$. In this case, Lemma 2.4 can be improved in the following way:

$$s - \lim_{\tau\to 0} (I + S_{it_0,\tau})^{-1} = (I + it_0 C)^{-1} P'. \tag{2.11}$$

For the proof, let us consider an approximate identity $\rho_n : \mathbb{R} \mapsto \mathbb{R}_+$. By Lemma 2.3, one has

$$\lim_{\tau\to 0} [\rho_n * (u, w_{\cdot,\tau})](t_0) = [\rho_n * (u, w_\cdot)](t_0) \qquad \text{for each } n = 1, 2, \ldots,$$

and by the equicontinuity of the functions $t \mapsto (u, w_{t,\tau})$ at t_0,

$$\lim_{n\to\infty} [\rho_n * (u, w_{\cdot,\tau})](t_0) = (u, w_{t_0,\tau}) \qquad \text{uniformly in } \tau > 0.$$

Then in the proof of the theorem we consider $Z_{t,n} = -t_0^{-1} S_{it_0, t/nt_0}$ for any $t \in \mathbb{R}$. By (2.11), one has

$$s - \lim_{n\to\infty} \left(t_0^{-1} - Z_{t,n}\right)^{-1} = \left(t_0^{-1} + iC\right)^{-1} P',$$

which leads to the result of Theorem 1.1 without subsequences (the exceptional set L has also disappeared).

Note added in proof (April 2005). For the product formula with projection, see also: P. Exner and T. Ichinose, 'A product formula related to quantum Zeno dynamics', *Ann. H. Poincaré*, to appear; arXiv:math-ph/0302060.

References

1. W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monogr. Math. 96 (Birkhäuser, Basel, 2001).

2. P. R. Chernoff, 'Product formulas, nonlinear semigroups, and addition of unbounded operators', *Mem. Amer. Math. Soc.* 140 (1974).

3. E. B. Davies: *One parameter semigroups* (Academic Press, London, 1980).

4. P. Exner: *Open quantum systems and Feynman integrals* (D. Reidel Publ. Co., Dordrecht, 1985).

5. J. Feldman: 'On the Schrödinger and heat equations for nonnegative potentials', *Trans. Amer. Math. Soc.* 108 (1963) 251–264.

6. E. Hille and R. S. Phillips: *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. 31 (Amer. Math. Soc., Providence, RI, 1957).

7. T. Kato: 'Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroup', *Topics in functional analysis* (ed. M. Kac, Academic Press, New York, 1978) 185–195.

8. M. Lapidus: 'Formule de moyenne et de produit pour les résolvantes imaginaires d'opérateurs auto-adjoints', *C. R. Acad. Sc. Paris Sér. A* 291 (1980) 451–454.

9. M. Matolcsi and R. Shvidkoy: 'Trotter's product formula for projections', *Arch. Math.* 81 (2003) 309–317.

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